

MAC-CPTM Situations Project

Situation 11: Faces of a Polyhedral Solid

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Prompt

In a 7th-grade mathematics class, in which the lesson was on classification of solids, the teacher held up a rectangular prism and asked the class how many "sides" there were. Two students responded, one with an answer of 12, the other with an answer of 6. The student with the answer of 6 was acknowledged as being correct, and the teacher continued with the lesson.

Commentary

Mathematics dictionary definitions of *solid* and *polyhedron* are not particularly clear. For example, a polyhedron is "a three-dimensional solid which consists of a collection of polygons, usually joined at their edges" (Weisstein, 2003) or "a solid with a surface composed of plane polygonal surfaces (*faces*)" (Nelson, 2003). A solid is "a closed, 3-dimensional figure" (Clapham & Nicholson, 2005) or "any limited portion of space bounded by surfaces" (Weisstein, 2003) or "(a three-dimensional geometric figure, e.g., a polyhedron or cone" (Nelson, 2003). Therefore, the title could use either the word *polyhedron* or *polyhedral solid*; we have chosen to use *polyhedral solid*. We are assuming that *polyhedron* refers only to the boundary points, which are the points on the polygonal faces, and that *solid* refers to both the boundary points and the interior points. By using *polyhedral solid*, we are referring to boundary points and interior points and assuming that the boundary points lie on polygonal faces.

The prompt raises the issue of how one refers to or names a mathematical entity, in this case, one of the constituting elements that form a prism, or more generally, a polyhedron. Focus 1 addresses the issue of the arbitrariness of conventional mathematical terms, and Focus 2 attends to differences between terms used for 2D figures and those used for 3D figures. Focus 3 uses Euler's formula, which relates the numbers of vertices, edges, and faces of a polyhedron, to determine the number of faces of the rectangular prism. The nature of a

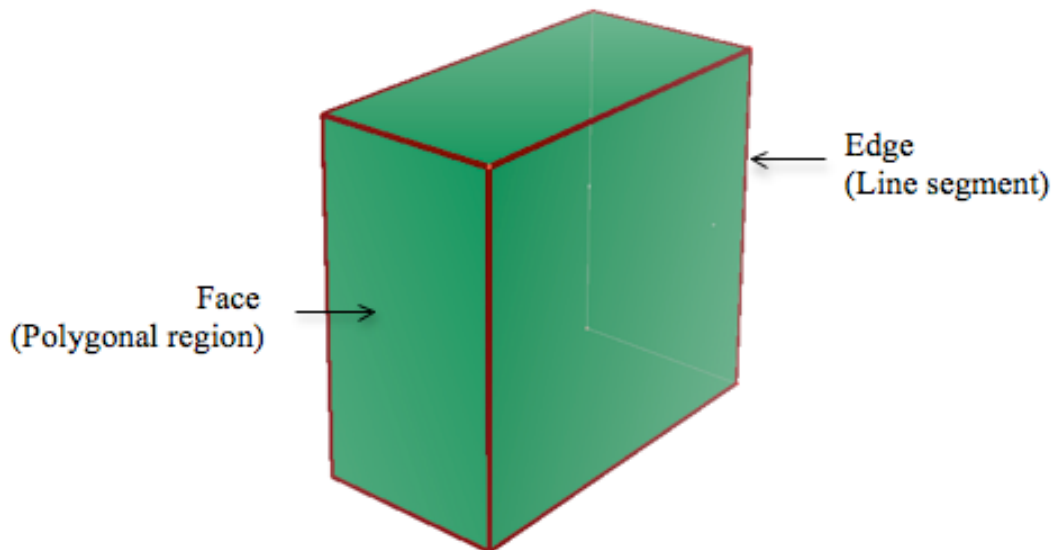
polygonal region (one type of bounded, planar region) addressed in Focus 4 offers one mathematical basis for understanding from where the student's answer of "12" might have come (other than from counting the 12 edges).

Mathematical Foci

Mathematical Focus 1

In mathematics, the words one uses to identify mathematical entities are conventional, but arbitrary.

The term *line segment* refers to the union of two distinct points and the set of collinear points between them. However, one could just as well use the term *line part*, if that were the accepted term for what typically refers to a *line segment*. Although, as the teacher did in the prompt, one could call one of the polygonal regions that constitute a polyhedron a *side* (or some other term such as *slab* or *dwizzle*), the typical term used for such a polygonal region in the context of a polyhedron is *face*. Using the convention that for a polyhedral solid, *edge* refers to a line segment and *face* refers to a polygonal region, a rectangular prism has 6 faces and 12 edges. The first student may have inferred that the teacher meant *edge* by the term *side*, whereas the second student may have inferred that the teacher meant *face* by the term *side*.



Mathematical Focus 2

Mathematical terms do not necessarily carry the same meaning when moving from one system or dimension to another. When moving between dimensional contexts a term may take on different meanings that are not communicated by the term alone.

There are some similarities and some differences between mathematical entities in 2D and those in 3D. Likewise, there are similarities and differences between the terms used to denote those entities.

Consider the terms used with polygons in 2D and the corresponding referents for those terms: *Vertex* refers to a point (a 0-dimensional object) and *side* refers to the infinite set of points on a line segment (a 1-dimensional object). The terms used with prisms (or, more generally, polyhedra) in 3D and their corresponding referents are *vertex* for a point (a 0-dimensional object), *edge* for the infinite set of points on a line segment (a 1-dimensional object), and *face* for the infinite set of points in a bounded, polygonal region (a 2-dimensional object).

If the term *side* were used in place of either *edge* or *face*, would it be used to refer to an edge or to a face? If one attempted to map the 2D terms directly onto the 3D objects, one might use *side* in the same way as in 2D (i.e., to denote a segment—what one usually calls an *edge*). But, one also could use *side* to denote one of the faces, given that one typically means *face* when referring to things like the north *side* of a building or the left *side* of a box. So, potential confusion could arise if what typically are called *faces* were called *sides*, because one also could use the word *side* to refer to one of the segments forming the polyhedron's edges. In fact, some definitions of *edge* use the word *side* interchangeably with *edge*, for example, “Edge: A LINE SEGMENT where two faces of a POLYHEDRON meet, also called a SIDE” (Weisstein, 2003) and others use *side* for *face*, for example, “Side: One of the lines joining adjacent vertices in a polygon. One of the faces in a polyhedron” (Clapham & Nicholson, 2005). Using *side* to denote a two-dimensional polygonal region (face) as well as a one-dimensional segment (edge) could be confusing. Hence, rather than *side*, the term *edge* is typically used to denote a segment that is the intersection of exactly two faces of a prism, and the term *face* is used to denote a polygonal region bounding a prism (or, more generally, a polyhedron).

Mathematical Focus 3

One can use Euler's formula (the “polyhedron formula”)¹ for the number of vertices, faces, and edges of a convex polyhedron to determine the number of faces of a rectangular prism if one knows the number of vertices and edges.

Although one could obtain 6 for the number of faces of a rectangular prism by counting, one also could use Euler's formula,

$$V - E + F = 2 \text{ or } E - V + 2 = F,$$

¹ The polyhedron formula given here is not the only formula referred to as *Euler's formula*. Other formulas attributed to Euler, such as $e^{i\pi} = -1$, may be more commonly thought of as Euler's formula.

which relates the numbers of vertices, edges, and faces of a polyhedron.² One could subtract the number of vertices (8) from the number of edges (12) and add 2 to get the number of faces (6).

Mathematical Focus 4

A polygonal region (a type of bounded, planar region) is a set of points that does not have two distinct “surfaces.”

The student who answered “12” may have been thinking that a rectangular prism has 12 faces: 6 faces on its “outside” and 6 faces on its “inside.” This is not surprising, given that transparent, plastic models of prisms have “thickness,” and one can see one surface on the outside of the model and a different surface on the inside. The fundamental question is, “Does a polygonal region have two ‘sides’?”

Although an observer of a polygonal region can be situated in either of two half-spaces formed by the plane in which the polygon lies, the polygonal region consists of only one set of points that is infinitely “thin.” There are not two distinct sets of points, one forming a polygonal region in one half-space and another forming a different polygonal region in the other half-space. A polyhedron’s face (what the teacher called a “side”) is the same set of points, regardless of the half-space from which it is viewed. Therefore, there are only 6 faces, not 12.

References

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² See Courant & Robbins, (1941, pp. 236–240) for a proof of Euler’s polyhedron formula. Also, Eppstein (2005) presents 19 proofs of Euler’s polyhedron formula.